Fast quantum algorithms for approximating some irreducible representations of groups

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Abstract

We consider the quantum complexity of estimating matrix elements of unitary irreducible representations of groups. For several finite groups including the symmetric group, quantum Fourier transforms yield efficient solutions to this problem. Furthermore, quantum Schur transforms yield efficient solutions for certain irreducible representations of the unitary group. Beyond this, we obtain poly(n)-time quantum algorithms for approximating matrix elements from all the irreducible representations of the alternating group A_n, and all the irreducible representations of polynomial highest weight of $U(n)$, $SU(n)$, and $SO(n)$. These quantum algorithms offer exponential speedup in worst case complexity over the fastest known classical algorithms. On the other hand, we show that average case instances are classically easy, and that the techniques analyzed here do not offer a speedup over classical computation for the estimation of group characters.

1 Introduction

Explicit representations of groups have many uses in physics, chemistry, and mathematics. All representations of finite groups and compact linear groups can be expressed as unitary matrices given an appropriate choice of basis[5]. This makes them natural candidates for implementation using quantum circuits. Here we show that polynomial size quantum circuits can implement:

- The irreducible representations of any finite group which has an efficient quantum Fourier transform. This includes the symmetric group S_n.
- The irreducible representations of the alternating group A_n.
- The irreducible representations of polynomial highest weight of the unitary $U(n)$, special unitary $SU(n)$, and special orthogonal $SO(n)$ groups.

Using these quantum circuits one can find a polynomially precise additive approximation to any matrix element of these representations by repeating a simple measurement called the Hadamard test, as described in section 2.

More precisely, for the finite groups S_n and A_n we obtain any matrix element of any irreducible representation to within $\pm\epsilon$ in time that scales polynomially in $1/\epsilon$ and n. For the Lie groups $U(n)$, $SU(n)$, and $SO(n)$ we obtain any matrix element of any irreducible representation of polynomial highest weight to within $\pm\epsilon$ in time that scales polynomially in $1/\epsilon$ and n. Because the representations considered are of exponentially large dimension, one cannot efficiently find these matrix elements by classically multiplying the matrices representing a set of generators. Note that, many computer science applications use multiplicative approximations. In this case, one computes an estimate $\tilde{x}$ of a quantity x with the requirement that $(1-\epsilon)x \leq \tilde{x} \leq (1-\epsilon)x$. The approximations obtained in this paper are all additive rather than multiplicative.

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	symmetric	braid
matrix elements	in BQP	BQP-complete [3, 37]
normalized characters	in BPP	DQC1-complete [33, 25]

Figure 1: The complexity results on the symmetric group refer arbitrary irreducible representations in Young's orthogonal form. The results on the braid group refer to the Jones-Wenzl representations, which give rise to Jones and HOMFLY polynomials. The complexity class DQC1 is the set of problems solvable in polynomial time on a one clean qubit computer. It is generally believed that one clean qubit computers are weaker than standard quantum computers but still capable of solving some problems outside of BPP.

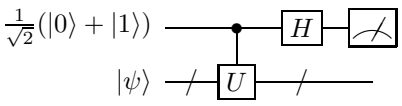
For some problems, the computational complexity of additive approximations can differ greatly from that of mulitplicative approximations[11, 2].

For exponentially large unitary matrices, the typical matrix element is exponentially small. Thus for average instances, a polynomially precise additive approximation provides almost no information. However, it is common that the worst case instances of a problem are hard whereas the average case instances are trivial. In section 5 I narrow down a class of potentially hard instances for the problem of additively approximating the matrix elements of the irreducible representations of the symmetric group to polynomial precision. I also present a classical randomized algorithm to estimate normalized characters of the symmetric group S_n to within $\pm\epsilon$ in $\mathrm{poly}(n, 1/\epsilon)$ time. (The character is normalized by dividing by the dimension of the representation, so that the character of the identity element of the group is 1.) Thus, the techniques described here for evaluating matrix elements of irreducible representations of groups on quantum computers do not provide an obvious quantum speedup for the evaluation of the characters of S_n.

Our results on the symmetric group relate closely to the quantum complexity of evaluating Jones polynomials and other topological invariants. Certain problems of approximating Jones and HOMFLY polynomials can be reduced to the approximation of matrix elements or characters of the Jones-Wenzl representation of the braid group, which is a q-deformation of certain irreducible representations of the symmetric group [3, 37, 33, 25]. Figure 1 compares the complexity of estimating matrix elements and characters of the Jones-Wenzl representation of the braid group to the complexity of the corresponding problems for the symmetric group. Exact complexity characterizations (*i.e.* completeness results) are not known for all of these problems, and the exact relationships between the complexity classes referenced in figure 1 are not rigorously known. Nevertheless, the results seem to suggest that in general the matrix elements are harder to approximate than the normalized characters, and that the Jones-Wenzl representation of braid group is computationally harder than the corresponding irreducible representations of the symmetric group.

2 Hadamard Test

The Hadamard test is a standard technique in quantum computation for approximating matrix elements of unitary transformations. Suppose we have an efficient quantum circuit implementing a unitary transformation U, and an efficient procedure for preparing the state $|\psi\rangle$. We can then approximate the real part of $\langle\psi| U |\psi\rangle$ using the following quantum circuit.



The probability of measuring $|0\rangle$ is

$$p_0 = \frac{1 + \mathrm{Re}(\langle\psi| U |\psi\rangle)}{2}.$$

Thus, one can obtain the real part of $\langle\psi| U |\psi\rangle$ to precision ϵ by making $O(1/\epsilon^2)$ measurements and counting what fraction of the measurement outcomes are $|0\rangle$. Similarly, if the control bit is instead initialized to

$\frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$, one can estimate the imaginary part of $\langle\psi|\,U\,|\psi\rangle$. Thus the problem of estimating matrix elements of unitary representations of groups reduces to the problem of implementing these representations with efficient quantum circuits.

3 Fourier Transforms

Let G be a finite group and let $\hat{G}$ be the set of all irreducible representations of G. We choose a basis for the representations such that for any $\rho \in \hat{G}$ and $g \in G$, g is represented by a $d_\rho \times d_\rho$ unitary matrix with entries $\rho_{i,j}(g)$. The quantum Fourier transform over G is the following unitary operator[29]

$$U_{\mathrm{FT}} = \sum_{g\in G} \sum_{\rho\in\hat{G}} \sum_{i,j=1}^{d_\rho} \sqrt{\frac{d_\rho}{|G|}} \rho_{i,j}(g)\,|\rho,i,j\rangle\,\langle g|\,.$$

Here $|g\rangle$ is a computational basis state (bitstring) indexing the element g of G. Similarly, $|\rho,i,j\rangle$ is three bitstrings, one indexing the element $\rho \in \hat{G}$, and two writing out the numbers i and j in binary. The standard discrete Fourier transform is the special case where G is a cyclic group.

The regular representation of any $g \in G$ is

$$U_g = \sum_{h\in G} |gh\rangle\,\langle h|\,.$$

A short calculation shows

$$U_{\mathrm{FT}} U_g U_{\mathrm{FT}}^{-1} = \sum_{\rho\in\hat{G}} \sum_{i,j=1}^{d_\rho} \sum_{i',j'=1}^{d_\rho} \delta_{j,j'} \rho_{i,i'}(g^{-1})\,|\rho,i,j\rangle\,\langle\rho,i',j'|\,.$$

In other words, by conjugating the regular representation of g with the quantum Fourier transform, one recovers the direct sum of all irreducible representations of g^{-1}.

Given an efficient quantum circuit implementing U_{FT} one can thus efficiently estimate any matrix element of any irreducible representation of G using the Hadamard test. Quantum circuits implementing the Fourier transform in polylog($|G|$) time are known for the symmetric group[8] and several other groups[28]. The matrix elements of the representations depend on a choice of basis. The bases used in quantum Fourier transforms are subgroup adapted (see [28]). In particular, the symmetric group Fourier transform described in [8] uses the Young-Yamanouchi basis, also known as Young's orthogonal form.

In section 8 we describe a more direct quantum circuit implementation of the irreducible representations of the symmetric group, which generalizes to yield efficient implementations for the alternating group.

4 Schur Transform

Let $\mathcal{H}$ be the Hilbert space of n d-dimensional qudits.

$$\mathcal{H} = (\mathbb{C}^d)^{\otimes n}.$$

We can act on this Hilbert space by choosing an element $u \in U(d)$ and applying it to each qudit.

$$|\psi\rangle \to u^{\otimes n}\,|\psi\rangle$$

We can also act on this Hilbert space by choosing an element $\pi \in S_n$ and correspondingly permuting the n qudits.

$$|\psi\rangle \to M_\pi\,|\psi\rangle$$

$u^{\otimes n}$ and M_π are reducible unitary nd-dimensional representations of $U(d)$ and S_n, respectively. These two actions on $\mathcal{H}$ commute.

The irreducible representations of S_n are in bijective correspondence with the partitions of n. Any partition of n into d parts indexes a unique irreducible representation of $U(d)$. $U(d)$ has infinitely many irreducible representations, so these partitions only index a special subset of them. As discussed in [6], there exists a unitary change of basis U_{Schur} such that

$$U_{\text{Schur}} M_\pi u^{\otimes n} U_{\text{Schur}}^{-1} = \bigoplus_\lambda \rho_\lambda(\pi) \otimes \nu_\lambda(u),$$

where λ ranges over all partitions of n into d parts.

As shown in [6], U_{Schur} can be implemented by a $\text{poly}(n, d)$ size quantum circuit. Thus, using the Hadamard test, one can efficiently obtain matrix elements of these representations of the symmetric and unitary groups.

5 Complexity of Symmetric Group Representations

As described in section 3, quantum computers can solve the following problem with probability $1 - \delta$ in $\text{poly}(n, 1/\epsilon, \log(1/\delta))$ time. Note that standard Young tableaux index the Young-Yamanouchi basis vectors, as discussed in section 8.1.

Problem 1: Approximate a matrix element in the Young-Yamanouchi basis of an irreducible representation for the symmetric group S_n.
Input: A Young diagram specifying the irreducible representation, a permutation from S_n, a pair of standard Young tableaux indicating the desired matrix element, and a polynomially small parameter ϵ.
Output: The specified matrix element to within $\pm\epsilon$.

It appears that no polynomial time classical algorithm for this problem is known. Due mainly to applications in quantum chemistry, many exponential time classical algorithms for the exact computation of entire matrices from representations of the symmetric group have been developed[22, 10, 38, 39, 14, 13, 31, 30]. There appears to be no literature on the computation or approximation of individual matrix elements of representations of S_n.

On the other hand, the precision of approximation achieved by the quantum algorithm is trivial for average instances. We can see this as follows. Let λ be a Young diagram of n boxes, let ρ_λ be the corresponding irreducible representation of S_n, and let d_λ be the dimension of ρ_λ. For any $\pi \in S_n$, the root mean square of the matrix elements of $\rho_\lambda(\pi)$ is

$$\text{RMS}(\rho_\lambda(\pi)) = \sqrt{\frac{1}{d_\lambda^2} \sum_{a,b \in B} |\langle a| \rho_\lambda(\pi) |b\rangle|^2},$$

where B is any complete orthonormal basis for the vector space on which ρ_λ acts. We see that

$$\sum_{a \in B} |\langle a| \rho_\lambda(\pi) |b\rangle|^2 = 1$$

since, by the unitarity of $\rho_\lambda(\pi)$, this is just the norm of $|b\rangle$. Thus,

$$\text{RMS}(\rho_\lambda(\pi)) = \sqrt{\frac{1}{d_\lambda^2} \sum_{b \in B} 1} = \frac{1}{\sqrt{d_\lambda}}. \tag{1}$$

The interesting instances of problem 1 are those in which d_λ is exponentially large. In these instances, the typical matrix element is exponentially small, by equation 1. Running the quantum algorithm yields polynomial precision, thus one could instead simply guess zero every time, with similar results.

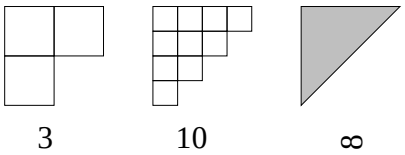


Figure 2: Here is a sequence of Young diagrams, such that as the number of boxes increases, the Young diagram converges asymptotically to some fixed shape, in this case a triangle.

That the average case instances are trivial does not mean that the algorithm is trivial. Hard problems that are trivial on average are a common occurrence. The most relevant example of this is the problem of estimating a knot invariant called the Jones polynomial. A certain problem of estimating the Jones polynomial of knots is BQP-complete[16, 3, 1]. The Jones polynomial algorithm is based on estimating matrix elements of certain representations of the braid group to polynomial precision. On average these matrix elements are exponentially small. Nevertheless, the BQP-hardness of the Jones polynomial problem shows that the worst-case instances are as hard as any problem in BQP.

By analogy to the results on Jones polynomials, one might ask ask whether problem 1 is BQP-hard. The existing proofs of BQP-hardness of Jones polynomial estimation rely on the fact that the relevant representations of the braid group are dense in the corresponding unitary group. Thus, one can construct a braid whose representation implements approximately the same unitary as any given quantum circuit. Furthermore, it turns out that the number of crossings needed to achieve a good approximation scales only polynomially with the number of quantum gates in the circuit. Unlike the braid group, the symmetric group is finite. Thus, no representation of it can be dense in a continuous group. Hence, if the problem of estimating matrix elements of the symmetric group is BQP-hard, the proof will have to proceed along very different lines than the BQP-hardness proof for Jones polynomials.

Lacking a hardness proof, the next best thing is to identify a class of instances in which the matrix elements are large enough to make the approximation nontrivial. As shown below, we can do this using the asymptotic character theory of the symmetric group. Note that we need not worry about the matrix elements being too large, because even if we know *a priori* that a given matrix element has magnitude 1, it could still be nontrivial to compute its sign.

Let π be a permutation in S_n, and let λ be a Young diagram of n boxes. The character

$$\chi_\lambda(\pi) = \mathrm{Tr}(\rho_\lambda(\pi))$$

is clearly independent of the basis in which ρ_λ is expressed. Furthermore, the character of a group element depends only on the conjugacy class of the group element, because for any representation ρ,

$$\mathrm{Tr}(\rho(hgh^{-1})) = \mathrm{Tr}(\rho(h)\rho(g)\rho(h)^{-1}) = \mathrm{Tr}(\rho(g)).$$

To understand the behavior of the characters of S_n as n becomes large, consider a sequence of Young diagrams $\lambda_1, \lambda_2, \lambda_3, \ldots$, where λ_n has n boxes. Suppose that the diagram λ_n, when scaled down by a factor of $1/\sqrt{n}$, converges to a fixed shape ω in the limit of large n, as illustrated in figure 2. Let d_{λ_n} be the dimension of the irreducible representation corresponding to Young diagram λ_n. Let π be a permutation in S_k. We can also consider π to be an element of S_n for any $n > k$ which leaves the remaining $n - k$ objects fixed. As shown by Biane[9],

$$\frac{\chi_{\lambda_n}(\pi)}{d_{\lambda_n}} = C_\pi(\omega) n^{-|\pi|/2} + O(n^{-|\pi|/2-1}). \tag{2}$$

Here $|\pi|$ denotes the minimum number of transpositions needed to obtain π. Note that these are general transpositions, not transpositions of neighbors. $C_\pi(\omega)$ is a constant that only depends on $\pi \in S_k$ and the shape ω. A precise definition of what it means for the sequence to converge to a fixed shape is given in [9], but for present purposes, the intuitive picture of figure 2 should be sufficient.

$\chi_{\lambda_n}(\pi)/d_{\lambda_n}$ is the average of the matrix elements on the diagonal of $\rho_{\lambda_n}(\pi)$. In the present setting, where π is fixed, $\chi_{\lambda_n}(\pi)/d_{\lambda_n}$ shrinks only polynomially with n. Thus polynomial precision is sufficient to provide nontrivial estimates of these matrix elements. Nevertheless, finding diagonal matrix elements of $\rho_{\lambda_n}(\pi)$ for fixed π and large n is not computationally hard. This is because, as discussed in section 8, the Young-Yamanouchi basis is subgroup adapted. Thus, for any π which leaves all bit the first k objects fixed, $\rho_{\lambda_n}(\pi)$ is a direct sum of irreducible representations of π in S_k. Because k is fixed, any irreducible representations of S_k has dimension $O(1)$ and can therefore be computed in $O(1)$ time by multiplying the matrices representing transpositions.

To produce a candidate class of hard instances of problem 1, we recall that the character $\chi_{\lambda_n}(\pi)$ depends only on the conjugacy class of π. Thus, we consider π' conjugate to π. Like $\pi \in S_n$, $\pi' \in S_n$ leaves at least $n-k$ objects fixed, and the representations $\chi_{\lambda_n}(\pi')$ have diagonal matrix elements with polynomially small average value. However, the objects left fixed by π' need not be $k+1, k+2, \ldots, n$. Indeed, π' can be chosen so that the object n is not left fixed, in which case $\rho_{\lambda_n}(\pi')$ cannot be written as the direct sum of irreducible representations of S_m for any $m < n$.

There is an additional simple way in which an instance of problem 1 can fail to be hard. Let $r(\pi)$ be the minimal number of transpositions of neighbors needed to construct the permutation π. If $r(\pi)$ is constant or logarithmic, then the matrix elements of the irreducible representations of π can be computed classically in polynomial time by direct recursive application of equation 13. For a class of hard instances of problem 1 I propose the following.

Hypothesis 1 *Let π be a permutation in S_n. We consider it to permute a series of objects numbered $1, 2, 3, \ldots, n$. Let $s(\pi)$ be the number of objects that π does not leave fixed. Let $l(\pi)$ be the largest numbered object that π does not leave fixed. Let $r(\pi)$ be the minimum number of transpositions of neighbors needed to construct π. Let λ be a Young diagram of n boxes, and let ρ_λ be the corresponding d_λ-dimensional irreducible representation of S_n. I propose the problems of estimating the diagonal matrix elements of $\rho_\lambda(\pi)$ such that $s(\pi) = O(1)$, $l(\pi) = \Omega(n)$, and $r(\pi) = \Omega(n)$ as a possible class of instances of problem 1 not solvable classically in polynomial time.*

Although this hypothesis contains many restrictions on π, it is clear that permutations satisfying all of these conditions exist. One simple example is the permutation that transposes 1 with n.

6 Characters of the Symmetric Group

Because characters do not depend on a choice of basis, the computational complexity of estimating characters is especially interesting. Hepler[24] showed that computing the characters of the symmetric group exactly is #P-hard. It is clear that an algorithm for efficiently approximating matrix elements of a representation can aid in approximating the corresponding character. Specifically, the quantum algorithm for problem 1 yields an efficient solution for the following problem.

Problem 2: Approximate a character for the symmetric group S_n.
Input: A Young diagram λ specifying the irreducible representation, a permutation π from S_n, and a polynomially small parameter ϵ.
Output: Let $\chi^\lambda(\pi)$ be the character, and let d_λ be the dimension of the irreducible representation. The output χ_{out} must satisfy $|\chi_{\text{out}} - \chi^\lambda(\pi)/d_\lambda| \leq \epsilon$ with high probability.

However, as we show in this section, problem 2 is efficiently solvable using only classical randomized computation. Thus the techniques used for problem 1 do not offer immediate benefit for problem 2. Although this is in some sense a negative result, it provides an interesting illustration of the difference in complexity between estimating individual matrix elements of representations and estimating the characters.

We can reduce problem 2 to problem 1 by sampling uniformly at random from the standard Young tableaux compatible with Young diagram λ. For each Young tableau sampled we estimate the corresponding diagonal matrix element of $\rho_\lambda(\pi)$, as described in problem 1. By averaging the diagonal matrix elements

for polynomially many samples, we obtain the normalized character to polynomial precision. The problem of sampling uniformly at random from the standard Young tableaux of a given shape is nontrivial but it has been solved. Greene, Nijenhuis, and Wilf proved in 1979 that their "hook-walk" algorithm produces the standard Young tableaux of any given shape with uniform probability[21]. Examination of [21] shows that the time needed by the hook-walk algorithm to produce a random standard Young tableaux compatible with a Young diagram of n boxes is upper bounded by $O(n^2)$.

By averaging over diagonal matrix elements we lose some information contained in the individual matrix elements. This observation gives the intuition that it should often be harder to estimate individual matrix elements of a representation than to estimate its trace. Jones polynomials provide an example in which this intuition is confirmed. As discussed in [33], computing the Jones polynomial of the trace closure of a braid reduces to computing the normalized character of a certain representation of the braid group. The problem of additively approximating this normalized character is only DQC1-complete. In contrast, the individual matrix elements of this representation yield the Jones polynomial of the plat closure of the braid and are BQP-complete to approximate. We see a very similar phenomenon in the symmetric group; problem 2 is is solvable by a randomized polynomial-time classical algorithm, whereas problem 1 is not, as far as we know.

To construct a classical algorithm for problem 2, first recall that the character of a given group element depends only on the element's conjugacy class. We can think of any $\pi \in S_n$ as acting on the set $\{1, 2, \ldots, n\}$. The sizes of the orbits of the elements of $\{1, 2, \ldots, n\}$ under repeated application of π form a partition of the integer n. For example, consider the permutation $\pi \in S_5$ defined by

$$\pi(1) = 2 \quad \pi(2) = 3 \quad \pi(3) = 1 \quad \pi(4) = 5 \quad \pi(5) = 4.$$

This divides the set $\{1, 2, 3, 4, 5\}$ into the orbits $\{1, 2, 3\}$ and $\{4, 5\}$. Thus it corresponds to the partition $(3, 2)$ of the integer 5. Two permutations in S_n are conjugate if and only if they correspond to the same partition. Thus, we can introduce the following notation. For any two partitions μ and λ of n define χ^λ_μ to be the irreducible character of S_n corresponding to the Young diagram of λ evaluated at the conjugacy class corresponding to μ.

To obtain an efficient classical solution to problem 2 we use the following theorem due to Roichman[32].

Theorem 1 (From [32]) *For any partitions $\mu = (\mu_1, \ldots, \mu_l)$ and $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n, the corresponding irreducible character of S_n is given by*

$$\chi^\lambda_\mu = \sum_\Lambda W_\mu(\Lambda)$$

where the sum is over all standard Young tableaux Λ of shape λ and

$$W_\mu(\Lambda) = \prod_{\substack{1 \le i \le k \\ i \notin B(\mu)}} f_\mu(i, \Lambda)$$

where $B(\mu) = \{\mu_1 + \ldots + \mu_r | 1 \le r \le l\}$ and

$$f_\mu(i, \Lambda) = \begin{cases} -1 & \text{box } i+1 \text{ of } \Lambda \text{ is in the southwest of box } i \\ 0 & i+1 \text{ is in the northeast of } i,\ i+2 \text{ is in the southwest of } i+1, \text{ and } i+1 \notin B(\mu) \\ 1 & \text{otherwise} \end{cases}$$

By using the hook walk algorithm we can sample uniformly at random from the standard Young tableaux Λ of shape λ. By inspection of theorem 1 we see that for each Λ sampled we can compute $W_\mu(\Lambda)$ classically in poly(n) time. By averaging the values of $W_\mu(\Lambda)$ obtained during the course of the sampling we can thus obtain a polynomially accurate additive approximation the the normalized character, thereby solving problem 2.

Some readers may notice that theorem 1 is similar in form to the much older and better-known Murnaghan-Nakayama rule. However, the Murnaghan-Nakayama rule is based on a sum over all "rim-hook tableaux" of shape λ (see [32]). It is not obvious how to sample uniformly at random from the rim-hook tableaux of a given shape. Thus, it is not obvious how to use the Murnaghan-Nakayame rule to obtain a probabilistic classical algorithm for problem 2.

7 Lie Groups

7.1 Introduction

Because $U(n)$, $SU(n)$ and $SO(n)$ are compact linear groups, all of their representations are unitary given the right choice of basis[5]. In section 4 we described how to efficiently approximate the matrix elements from certain unitary irreducible representation of $U(n)$. Here we present a more direct approach to this problem, which can handle a larger set of representations of $U(n)$ and also extends to some other compact Lie groups: $SU(n)$ and $SO(n)$.

$U(n)$, $SU(n)$, and $SO(n)$ are subgroups of $GL(n)$, the group of all invertible $n \times n$ matrices. All of the irreducible representations of $U(n)$ and $SU(n)$ can be obtained by restricting the irreducible representations of $GL(n)$ to these subgroups. The best classical algorithms for computing irreducible representations of $GL(n)$ and $U(n)$ appear to be those of [12] and [20]. These classical algorithms work by manipulating matrices whose dimension equals the dimension of the representation. Thus, they do not provide a polynomial time algorithm for computing matrix elements from representations whose dimension is exponentially large. The implementation of irreducible representations of $SO(3)$ and $SU(2)$ by quantum circuits has been studied previously by Zalka[40].

7.2 Gel'fand-Tsetlin representation of $U(n)$

The irreducible representations of the Lie group $U(n)$ are most easily described in terms of the corresponding Lie algebra $u(n)$. It is not necessary here delve into the theory of Lie groups and Lie algebras, but those who are interested can see [19]. For now it suffices to say that $u(n)$ is the set of all antihermitian $n \times n$ matrices, and for any $u \in U(n)$ there exists $h \in u(n)$ such that $u = e^h$. Given any representation $a : u(n) \to u(m)$ one can construct a representation $A : U(n) \to U(m)$ as follows. For any $u \in U(n)$ find a corresponding $h(u) \in u(n)$ such that $e^h = u$, and set $A(u) = e^{a(h(u))}$. If a is an antihermitian representation of $u(n)$ then A is a unitary representation of $U(n)$. Furthermore, it is clear that A is irreducible if and only if a is irreducible.

It turns out that the irreducible representations of the algebra $gl(n)$ of all $n \times n$ complex matrices remain irreducible when restricted to the subalgebra $u(n)$. Furthermore, all of the irreducible representations of $u(n)$ are obtained this way. Let E_{ij} be the $n \times n$ matrix with all matrix elements equal to zero except for the matrix element in row i, column j, which is equal to one. The set of all n^2 such matrices forms a basis over $\mathbb{C}$ for $gl(n)$. Thus to describe a representation of $gl(n)$ it suffices to describe its action on each of the E_{ij} matrices.

As described in chapter 18, volume 3 of [35], explicit matrix representations of $gl(n)$ were constructed by Gel'fand and Tsetlin. (See also [18].) In their construction, one thinks of the representation as acting on the formal span of a set of combinatorial objects called Gel'fand patterns. The Gel'fand-Tsetlin representations of $E_{p,p-1}$ and $E_{p-1,p}$ are sparse and simple to compute for all $p \in \{2, 3, \ldots, n\}$. This property makes the Gel'fand-Tsetlin representations particularly useful for quantum computation.

A Gel'fand pattern of width n consists of n rows of integers[1]. The j^{th} row (from bottom) has j entries $m_{1,j}, m_{2,j}, \ldots, m_{j,j}$. (Note that, in contrast to matrix elements, the subscripts on the entries of Gel'fand patterns conventionally indicate column first, then row.) These entries must satisfy

$$m_{j,n+1} \geq m_{j,n} \geq m_{j+1,n+1}.$$

Gel'fand patterns are often written out diagrammatically. For example the Gel'fand pattern of width 3 with rows

$$\begin{array}{lll} m_{1,3} = 4 & m_{2,3} = 1 & m_{3,3} = 0 \\ m_{1,2} = 3 & m_{2,2} = 0 & \\ m_{1,1} = 2 & & \end{array}$$

[1]Some sources omit the top row, as it is left unchanged by the action of the representation.

is represented by the diagram

$$\begin{pmatrix} 4 & & 1 & & 0 \\ & 3 & & 0 & \\ & & 2 & & \end{pmatrix}.$$

This notation has the advantage that the entries that appear directly to the upper left and upper right of a given entry form the upper and lower bounds on the values that entry is allowed to take.

We call the top row of a Gel'fand pattern its weight[2]. To each weight of width n corresponds one irreducible representation of $gl(n)$. This irreducible representation acts on the formal span of all Gel'fand patterns with that weight (of which there are always finitely many). To describe the action of the representation of $gl(n)$ on these patterns let

$$l_{p,q} = m_{p,q} - p \tag{3}$$

$$a^j_{p-1} = \left| \frac{\prod_{i=1}^{p}(l_{i,p} - l_{j,p-1}) \prod_{i=1}^{p-2}(l_{i,p-2} - l_{j,p-1} - 1)}{\prod_{i \neq j}(l_{i,p-1} - l_{j,p-1})(l_{i,p-1} - l_{j,p-1} - 1)} \right|^{1/2} \tag{4}$$

$$b^j_{p-1} = \left| \frac{\prod_{i=1}^{p}(l_{i,p} - l_{j,p-1} + 1) \prod_{i=1}^{p-2}(l_{i,p-2} - l_{j,p-1})}{\prod_{i \neq j}(l_{i,p-1} - l_{j,p-1})(l_{i,p-1} - l_{j,p-1} + 1)} \right|^{1/2}. \tag{5}$$

Let M be a Gel'fand pattern and let M_p^{+j} be the Gel'fand pattern obtained from M by replacing $m_{j,p}$ with $m_{j,p} + 1$. Similarly, let M_p^{-j} be the Gel'fand pattern in which $m_{j,p}$ has been replaced with $m_{j,p} - 1$. The representation $a_{\vec{m}}$ of $gl(n)$ corresponding to weight $\vec{m} \in \mathbb{Z}^n$ is defined by the following rules[3], known as the Gel'fand-Tsetlin formulas.

$$a_{\vec{m}}(E_{p-1,p})M = \sum_{j=1}^{p-1} a^j_{p-1} M^{+j}_{p-1} \tag{6}$$

$$a_{\vec{m}}(E_{p,p-1})M = \sum_{j=1}^{p-1} b^j_{p-1} M^{-j}_{p-1} \tag{7}$$

$$a_{\vec{m}}(E_{p,p})M = \left(\sum_{i=1}^{p} m_{i,p} - \sum_{j=1}^{p-1} m_{j,p-1} \right) M \tag{8}$$

These formulas give implicitly a representation for all of $gl(n)$, because any E_{ij} can be obtained from operators of the form $E_{p-1,p}$ and $E_{p,p-1}$ by using the commutation relation $[E_{ik}, E_{kl}] = E_{il}$. By restricting the representation $a_{\vec{m}}$ to antihermitian subalgebra of $gl(n)$ and taking the exponential, one obtains an irreducible group representation $A_{\vec{m}} : U(n) \to U(d_{\vec{m}})$, where $d_{\vec{m}}$ is the number of Gel'fand patterns with weight $\vec{m}$.

It should be noted that some references claim that the set of allowed weights for representations of $GL(n)$ is $\mathbb{N}^n$, whereas others identify, as we do, $\mathbb{Z}^n$ as the allowed set of weights. The reason for this is that irreducible representations of $GL(n)$ in which the entries $m_{n,1}, m_{n,2}, \ldots, m_{n,n}$ of the weight are all nonnegative are polynomial invariants[26]. That is, for any $g \in GL(n)$ and any $\vec{m} \in \mathrm{N}^n$, each matrix element of the representation $\rho_{\vec{m}}(u)$ is a polynomial function of the n^2 matrix elements of u. The representations involving negative weights are called holomorphic representations, and many sources choose to neglect them. In the case that $\vec{m} \in \mathbb{N}^n$, the Gel'fand diagrams of width n bijectively correspond to the semistandard Young tableaux of n rows (*cf.* [15], pg. 517).

[2]It is actually the *highest* weight of the representation[35], but for brevity I just call it the weight throughout this paper.

[3]Warning: [35] contains a misprint, in which the sums in equations 6 and 7 are taken up to $j = p$ instead of $j = p - 1$.

7.3 Quantum Algorithm for U(n)

In this section we obtain an efficient quantum circuit implementation of any irreducible representation of $U(n)$ in which the entries $m_{1,n}, \ldots, m_{n,n}$ of the highest weight are all at most polynomially large. The dimension of such representations can grow exponentially with n. Unlike the Schur transform, the method here does not require $m_{1,n}, \ldots, m_{n,n}$ to be nonnegative. We start by finding a quantum circuit implementing the Gel'fand-Tsetlin representation of an $n \times n$ unitary matrix of the form

$$u_0 = \begin{bmatrix} u_{11} & u_{12} & & & & \\ u_{21} & u_{22} & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \end{bmatrix},$$

where all off-diagonal matrix elements not shown are zero. After that we describe how to extend the construction to arbitrary $n \times n$ unitaries.

For a given weight $\vec{m} \in \mathbb{Z}^n$ we wish to implement the corresponding representation $A_{\vec{m}}(u_0)$ with a quantum circuit. To do this, we first find an $n \times n$ Hermitian matrix H_0 such that $e^{iH_0} = u_0$. It is not hard to see that H_0 can be computed in polynomial time and takes the form

$$H_0 = \begin{bmatrix} h_{11} & h_{12} & & & & \\ h_{12}^* & h_{22} & & & & \\ & & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & \end{bmatrix}.$$

Thus,

$$H_0 = h_{11}E_{11} + h_{12}E_{12} + h_{12}^* h_{21} + h_{22}E_{22}. \tag{9}$$

Hence,

$$a_{\vec{m}}(H_0) = h_{11}a_{\vec{m}}(E_{11}) + h_{12}a_{\vec{m}}(E_{12}) + h_{12}^* a_{\vec{m}}(E_{21}) + h_{22}a_{\vec{m}}(E_{22}). \tag{10}$$

To implement $A_{\vec{m}}(u_0)$ with a quantum circuit, we think of $a_{\vec{m}}(H_0)$ as a Hamiltonian and simulate the corresponding unitary time evolution $e^{-ia_{\vec{m}}(H_0)t}$ for $t = -1$. The Hamiltonian $a_{\vec{m}}(H_0)$ has exponentially large dimension in the cases of computational interest. However, examination of equation 9 shows that H_0 is a linear combination of operators of the form $E_{p,p-1}$ and $E_{p-1,p}$. Thus, by the Gel'fand-Tsetlin rules of section 7.2, $a_{\vec{m}}(H_0)$ is sparse and that its individual matrix elements are easy to compute. Under this circumstance, one can use the general method for simulating sparse Hamiltonians proposed in [4].

Define row-sparse Hamiltonians to be those in which each row has at most polynomially many nonzero entries. Further, define row-computable Hamiltonians to be those such that there exists a polynomial time algorithm which, given an index i, outputs a list of the nonzero matrix elements in row i and their locations. Clearly, all row computable Hamiltonians are row-sparse. As shown in [4], the unitary e^{-iHt} induced by any row-computable Hamiltonian can be simulated in polynomial time provided that the spectral norm $\|H\|$ and the time t are at most polynomially large. We have already noted that $a_{\vec{m}}(H_0)$ is row-computable. $a_{\vec{m}}(H_0)$ is row sparse, and because we are considering only polynomial highest weight, the entries of the Gel'fand patterns, and hence the matrix elements of $a_{\vec{m}}(H_0)$ are only polynomially large. Thus, by Gershgorin's circle theorem $\|a_{\vec{m}}(H_0)\|$ is at most poly(n).

Having shown that a quantum circuit of poly(n) gates can implement the Gel'fand-Tsetlin representation of an $n \times n$ unitary of the form u_0, the remaining task is to extend this to arbitrary $n \times n$ unitaries. Examination of the preceding construction shows that it works just the same for any unitary of the form

$$u_p = \mathbb{1}_p \oplus u \oplus \mathbb{1}_{n-p-2},$$

where $\mathbb{1}_p$ denotes the $p \times p$ identity matrix and u is a 2×2 unitary. Corresponding to u_p is again an antihermitian matrix of the form

$$H_p = 0_p \oplus h \oplus 0_{n-p-2}$$

where 0_p is the $p \times p$ matrix of all zeros and h is a 2×2 antihermitian matrix such that $e^h = u$. The only issue to worry about is whether $\|a_{\vec{m}}(H_p)\|$ is at most poly(n). By symmetry, one expects that $\|a_{\vec{m}}(H_p)\|$ should be independent of p. However, this is not obvious from examination of equations 3 through 8. Nevertheless, it is true, as shown in appendix A. Thus, the norm is no different than in the $p = 0$ case, *i.e.* H_0.

By concatenating the quantum circuits implementing $A_{\vec{m}}(u_1), A_{\vec{m}}(u_2), \ldots, A_{\vec{m}}(u_L)$, one can implement $A_{\vec{m}}(u_1 u_2 \ldots u_L)$. We next show that any $n \times n$ unitary can be obtained as a product of poly(n) matrices, each of the form u_p, thus showing that the quantum algorithm is completely general and always runs in polynomial time.

For any 2×2 matrix M, let $\mathcal{E}(M, i, j)$ be the $n \times n$ matrix in which M acts on the i^{th} and j^{th} basis vectors. In other words, the k, l matrix element of $\mathcal{E}(M, i, j)$ is

$$\mathcal{E}(M,i,j)_{kl} = \begin{cases} M_{11} & \text{if } k = i \text{ and } l = i \\ M_{12} & \text{if } k = i \text{ and } l = j \\ M_{21} & \text{if } k = j \text{ and } l = i \\ M_{22} & \text{if } k = j \text{ and } l = j \\ \delta_{kl} & \text{otherwise} \end{cases} .$$

Thus

$$u_p = \mathcal{E}\left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, m+1, m+2 \right).$$

Next note that,

$$\mathcal{E}\left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, m+1, m+3 \right) =$$
$$\mathcal{E}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, m+2, m+3 \right) \mathcal{E}\left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, m+1, m+2 \right) \mathcal{E}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, m+2, m+3 \right).$$

Thus the matrix

$$\mathcal{E}\left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, m+1, m+3 \right)$$

is obtained as a product of three matrices of the form u_p. By repeating this conjugation process, one can obtain

$$\mathcal{E}\left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, i, j \right) \tag{11}$$

for arbitrary i, j as a product of one matrix of the form

$$\mathcal{E}\left(\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, p+1, p+2 \right)$$

for some p and at most $O(n)$ matrices of the form

$$\mathcal{E}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, q+1, q+2 \right)$$

with various q. A matrix of the form shown in equation 11 is called a two-level unitary. As shown in section 4.5.1 of [29], any $n \times n$ unitary is obtainable as a product of poly(n) two-level unitaries. Thus we obtain $A_{\vec{m}}(U)$ for any $n \times n$ unitary U using poly(n) quantum gates. One can then obtain any matrix element of $A_{\vec{m}}(U)$ to precision $\pm\epsilon$ by repeating the Hadamard test $O(1/\epsilon^2)$ times.

7.4 Special Orthogonal Group

The special orthogonal group $SO(n)$ consists of all $n \times n$ real orthogonal matrices with determinant equal to one. The irreducible representations of $SO(n)$ are closely related to those of $U(n)$ and can also be expressed unitarily using a Gel'fand-Tsetlin basis. As discussed in chapter 18, volume 3 of [35], the nature of the representations of $SO(n)$ depends on whether n is even or odd. Following [35] and [18], we therefore introduce an integer k and consider $SO(2k+1)$ and $SO(2k)$ separately.

The irreducible representations of $SO(2k+1)$ are in bijective correspondence with the set of allowed weight vectors $\vec{m}$ consisting of k entries, each of which is an integer or half-integer. Furthermore, the entries must satisfy

$$m_{1,n} \geq m_{2,n} \geq \ldots \geq m_{k,n} \geq 0.$$

The irreducible representations of $SO(2k)$ correspond to the weight vectors $\vec{m}$ with $k-1$ entries, each of which must be an integer or half integer, and which must satisfy

$$m_{1,n} \geq m_{2,n} \geq \ldots \geq m_{k-1,n} \geq |m_{k,n}|.$$

As in the case of $U(n)$, the set of allowed Gel'fand patterns is determined by rules for how a row can compare to the one above it. For $SO(n)$ these rules are slightly more complicated, and the rule for the j^{th} row depends on whether j is odd or even. Specifically the even rule for $j = 2k$ is

$$m_{1,2k+1} \geq m_{1,2k} \geq m_{2,2k+1} \geq m_{2,3k} \geq \ldots \geq m_{k,2k+1} \geq m_{k,2k} \geq -m_{k,2k-1},$$

and the odd rule for $j = 2k-1$ is

$$m_{1,2k} \geq m_{1,2k-1} \geq m_{2,2k} \geq m_{2,2k-1} \geq \ldots \geq m_{k-1,2k} \geq m_{k-1,2k-1} \geq |m_{k,2k}|.$$

The Lie algebra $so(n)$ corresponding to the Lie group $SO(n)$ is the algebra of all antisymmetric $n \times n$ matrices. For any $G \in SO(n)$ there exists a $g \in so(n)$ such that $e^g = G$. The Lie algebra $so(n)$ is the space of all $n \times n$ real traceless antisymmetric matrices. Thus it is spanned by operators of the form

$$I_{k,i} = E_{i,k} - E_{k,i} \quad 1 \leq i < k \leq n.$$

We can fully specify a representation of $so(n)$ by specifying the representations of the operators of the form $I_{q+1,q}$ because these generate $so(n)$. That is, any element of $so(n)$ can be obtained as a linear combination of commutators of such operators. The Gel'fand-Tsetlin representation $b_{\vec{m}}$ of these operators depends on whether q is even or odd, and is given by the following formulas.

$$\begin{aligned}
A^j_{2p}(M) &= \frac{1}{2}\left|\frac{\prod_{r=1}^{p-1}\left[(l_{r,2p-1}-\frac{1}{2})^2-(l_{j,2p}+\frac{1}{2})^2\right]\prod_{r=1}^{p}\left[(l_{r,2p+1}-\frac{1}{2})^2-(l_{j,2p}+\frac{1}{2})^2\right]}{\prod_{r\neq j}(l^2_{r,2p}-l^2_{j,2p})(l^2_{r,2p}-(l_{j,2p}+1)^2)}\right|^{1/2} \\
B^j_{2p+1}(M) &= \left|\frac{\prod_{r=1}^{p}(l^2_{r,2p}-l^2_{j,2p+1})\prod_{r=1}^{p+1}(l^2_{r,2p+2}-l^2_{j,2p+1})}{l^2_{j,2p+1}(4l^2_{j,2p+1}-1)\prod_{r\neq j}(l^2_{r,2p+1}-l^2_{j,2p+1})(l^2_{j,2p+1}-(l_{r,2p+1}-1)^2)}\right|^{1/2} \\
C_{2p}(M) &= \frac{\prod_{r=1}^{p} l_{r,2p}\prod_{r=1}^{p+1} l_{r,2p+2}}{\prod_{r=1}^{p} l_{r,2p+1}(l_{r,2p+1}-1)} \\
b_{\vec{m}}(I_{2p+1,2p})M &= \sum_{j=1}^{p} A^j_{2p}(M)M^{+j}_{2p} - \sum_{j=1}^{p} A^j_{2p}(M_2p^{-j})M^{-j}_{2p} \\
b_{\vec{m}}(I_{2p+2,2p+1})M &= \sum_{j=1}^{p} B^j_{2p+1}(M)M^{+j}_{2p+1} - \sum_{j=1}^{p} B^j_{2p+1}(M^{-j}_{2p+1})M^{-j}_{2p+1} + iC_{2p}(M)M
\end{aligned}$$

By applying these rules to the set of allowed Gel'fand patterns described above one obtains the irreducible representations of the algebra $so(n)$. By exponentiating these, one then obtains the irreducible representations of the group $SO(n)$. Thus the quantum algorithm for approximating the matrix elements of the irreducible representations of $SO(n)$ is analogous to that for $U(n)$.

7.5 Special Unitary Group

The irreducible representations of $SU(n)$ can be easily constructed from the irreducible representations of $U(n)$, using the following facts taken from chapter 10 of [7]. The representations of $U(n)$ can be partitioned into a set of equivalence classes of projectively equivalent representations. Two representations of $U(n)$ with weights $\vec{l} = (l_1, l_2, \ldots, l_n)$ and $\vec{m} = (m_1, m_2, \ldots, m_n)$ are projectively equivalent if and only if there exists some integer s such that $m_i = l_i + s$ for all $1 \leq i \leq n$. Any irreducible representation of $U(n)$ remains irreducible when restricted to $SU(n)$. Furthermore, by choosing one representative from each class of projectively equivalent representations of $U(n)$ and restricting to $SU(n)$ one obtains a complete set of inequivalent irreducible representations of $SU(n)$. The Lie algebra $su(n)$ corresponding to the Lie group $SU(n)$ is easily characterized; it is the space of all traceless $n \times n$ antihermitian matrices. Thus the matrix elements of the irreducible representations of $SU(n)$ are obtained by essentially the same quantum algorithm given for $U(n)$ in section 7.3.

7.6 Characters of Lie Groups

As always, an algorithm for approximating matrix elements immediately gives us an algorithm for approximating the normalized characters. However, the characters of $U(n)$, $SU(n)$, and $SO(n)$ are classically computable in poly(n) time. As discussed in [17], the characters of any compact Lie group are given by the Weyl character formula. In general this formula may involve sums of exponentially many terms. However, in the special cases of $U(n)$, $SU(n)$, and $SO(n)$ the formula reduces to simpler forms[17], given below.

Because characters depend only on conjugacy class, the character $\chi_{\vec{m}}(u)$ depends only on the eigenvalues of u. For $u \in U(n)$ let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues. Let $\vec{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$ be the weight of a representation of $U(n)$. Let

$$l_i = m_i + n - i \tag{12}$$

for each $i \in \{1, 2, \ldots, n\}$. The character of the representation of weight $\vec{m}$ is

$$\chi_{\vec{m}}^{U(n)}(u) = \frac{\det A}{\det B}$$

where A and B are the following $n \times n$ matrices

$$\begin{aligned} A_{ij} &= \lambda_i^{l_j} \\ B_{ij} &= \lambda_i^{n-j}. \end{aligned}$$

This formula breaks down if u has a degenerate spectrum. However, the value of the character for degenerate u can be obtained by taking the limit as some eigenvalues converge to the same value. As shown in [36], one can obtain the dimension $d_{\vec{m}}$ of the representation corresponding to a given weight $\vec{m}$ by calculating $\lim_{u \to \mathbb{1}} \chi_{\vec{m}}(u)$. Specifically, by choosing $\lambda_j = e^{ij\epsilon}$ for each $1 \leq j \leq n$ and taking the limit as $\epsilon \to 0$ one obtains

$$d_{\vec{m}} = \frac{\prod_{i<j}(l_j - l_i)}{\prod_{i<j}(j - i)},$$

where l_i is as defined in equation 12.

As discussed in section 7.5, the irreducible representations of $SU(n)$ are restrictions of irreducible representations of $U(n)$, therefore the characters of $SU(n)$ are given by the same formula as the characters of $U(n)$.

$SO(n)$ consists of real matrices. The characteristic polynomials of these matrices have real coefficients, and thus their roots come in complex conjugate pairs. Thus, the eigenvalues of an element $g \in SO(2k+1)$ take the form

$$\lambda_1, \lambda_2, \ldots, \lambda_k, 1, \lambda_1^*, \lambda_2^*, \ldots, \lambda_k^*,$$

and for $g \in SO(2k)$, the eigenvalues take the form

$$\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda_1^*, \lambda_2^*, \ldots, \lambda_k^*.$$

As discussed in [17], the characters of the special orthogonal group are given by

$$\chi_{\vec{m}}^{SO(2k+1)}(g) = \frac{\det C}{\det D}$$

and

$$\chi_{\vec{m}}^{SO(2k)}(g) = \frac{\det E + \det F}{\det G}$$

where C and D are the following $k \times k$ matrices

$$\begin{aligned} C_{ij} &= \lambda_j^{m_i+n-i+1/2} - \lambda_j^{-(m_i+n-i+1/2)} \\ D_{ij} &= \lambda_j^{n-i+1/2} - \lambda_j^{-(n-i+1/2)} \end{aligned}$$

and E, F, G are the following $(k-1) \times (k-1)$ matrices

$$\begin{aligned} E_{ij} &= \lambda_j^{l_i} + \lambda_j^{-l_i} \\ F_{ij} &= \lambda_j^{l_i} - \lambda_j^{-l_i} \\ G_{ij} &= \lambda_j^{n-i} + \lambda_j^{-(n-i)}, \end{aligned}$$

where l_i is as defined in equation 12.

As with $U(n)$, the character of any element with a degenerate spectrum can be obtained by taking an appropriate limit.

7.7 Open Problems Regarding Lie groups

The quantum circuits presented in the preceeding sections efficiently implement the irreducible representations of $U(n)$, $SU(n)$, and $SO(n)$ that have polynomial highest weight and polynomial n. It is an interesting open problem to implement irreducible representations with quantum circuits that scale polynomially in the number of digits used to specify the highest weight. Alternatively, one could try to implement an Schur transform to handle exponential highest weight, which is also an open problem. It is even concievable that Schur-like transforms could be efficiently implemented for exponential n. That is, there could exist a quantum circuit of polylog(n) gates implementing a unitary transform V such that for any $U \in U(n)$, VUV^{-1} is a direct sum of irreducible representations of U. Of course, if n is exponentially large, than we cannot have an explicit description of U, rather the group element U could itself be defined by a quantum circuit.

A completely different open problem is presented by the symplectic group. Having constructed quantum circuits for $SO(n)$ and $SU(n)$, the symplectic group is the only "classical" Lie group remaining to be analyzed. Thus it is natural to ask whether its irreducible representations can be efficiently implemented by quantum circuits. Two different groups can go by the name symplectic group depending on the reference. Connected non-compact simple Lie groups have no nontrivial finite-dimensional unitary representations (see [7], theorem 8.1.2). This applies to one of the groups that goes by the name of symplectic. On the other hand, the irreducible representations of the compact symplectic group seem promising for implementation by quantum circuits. The main task seems to be finding a basis for these representations that is subgroup adapted and makes the representations unitary. A non-unitary subgroup-adapted basis is given in [27].

8 Alternating Group

In section 3, we described a method to approximate matrix elements of the irreducible representations of the symmetric group using the symmetric group quantum Fourier transform. Here we take a more direct approach to this problem, which extends to the alternating group. To do this we must first explicitly describe the Young-Yamanouchi representation of the symmetric group.

$$\rho_{\ydiagram{3,1}}(\sigma_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{matrix} \ytableaushort{123,4} \\ \ytableaushort{124,3} \\ \ytableaushort{134,2} \end{matrix} \qquad \rho_{\ydiagram{3,1}}(\sigma_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{matrix} \ytableaushort{123,4} \\ \ytableaushort{124,3} \\ \ytableaushort{134,2} \end{matrix}$$

$$\rho_{\ydiagram{3,1}}(\sigma_3) = \begin{bmatrix} -\frac{1}{3} & \frac{\sqrt{8}}{3} & 0 \\ \frac{\sqrt{8}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \ytableaushort{123,4} \\ \ytableaushort{124,3} \\ \ytableaushort{134,2} \end{matrix}$$

Figure 3: The above matrices are irreducible representations in the Young-Yamanouchi basis with Young diagram $\ydiagram{3,1}$. Here σ_i is the permutation in S_4 that swaps i with $i+1$.

8.1 Young-Yamanouchi Representation

For a given Young diagram λ, let $\mathcal{V}_\lambda$ be the vector space formally spanned by all standard Young tableaux compatible with λ. For example, if

$$\lambda = \ydiagram{2,1,1}$$

then $\mathcal{V}_\lambda$ is the 3-dimensional space consisting of all formal linear combinations of

$$\ytableaushort{14,2,3} \quad , \quad \ytableaushort{13,2,4} \quad , \text{ and } \quad \ytableaushort{12,3,4} \quad .$$

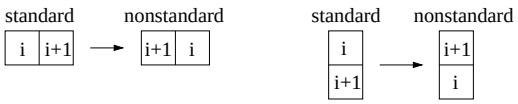
For any given Young diagram λ, the corresponding irreducible representation in the Young-Yamanouchi basis is a homomorphism ρ_λ from S_n to the group of orthogonal linear transformations on $\mathcal{V}_\lambda$. It is not easy to directly compute $\rho_\lambda(\pi)$ for an arbitrary permutation π. However, it is much easier to compute the representation of a transposition of neighbors. That is, we imagine the elements of S_n as permuting a set of objects $1, 2, \ldots, n$, arranged on a line. A neighbor transposition σ_i swaps objects i and $i+1$. It is well known that the set $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ generates S_n.

The matrix elements for the Young-Yamanouchi representation of transpositions of neighbors can be obtained using a single simple rule: Let Λ be any standard Young tableau compatible with Young diagram λ then

$$\rho_\lambda(\sigma_i)\Lambda = \frac{1}{\tau_i^\Lambda}\Lambda + \sqrt{1 - \frac{1}{(\tau_i^\Lambda)^2}}\Lambda', \tag{13}$$

where Λ' is the Young tableau obtained from Λ by swapping boxes i and $i+1$, and τ_i^Λ is the axial distance from box $i+1$ to box i. That is, we are allowed to hop vertically or horizontally to nearest neighbors, and τ is the number of hops needed to get from box $i+1$ to box i, where going down or left counts as $+1$ hop and going up or right counts as -1 hop. To illustrate the use of equation 13, some examples are given in figure 3.

In certain cases, starting with a standard Young tableau and swapping boxes i and $i+1$ does not yield a standard Young tableau, as illustrated below.



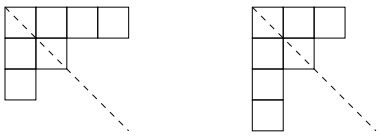


Figure 4: To obtain the conjugate $\hat{\lambda}$ of Young diagram λ, reflect λ about its diagonal. In other words the number of boxes in the i^{th} column of $\hat{\lambda}$ is equal to the number of boxes in the i^{th} row of λ.

Some thought shows that all such cases are of one of the two types shown above. In both of these types, the axial distance is ± 1. By equation 13, the coefficient on the invalid Young tableau is $\sqrt{1 - \frac{1}{(\pm 1)^2}} = 0$. Thus the representation lies strictly within the space of standard Young tableaux.

8.2 Direct Quantum Algorithm for S_n

We can directly implement the irreducible representations of S_n by first decomposing the given permutation into a product of transposition of neighbors. The classical bubblesort algorithm achieves this efficiently. For any permutation in S_n, it yields a decomposition consisting of at most $O(n^2)$ transpositions. As seen in the previous section, the Young-Yamanouchi representation of any transposition is a direct sum of 2×2 and 1×1 blocks, and the matrix elements of these blocks are easy to compute. As shown in [4], any unitary with these properties may be implemented by a quantum circuit with polynomially many gates. By concatenating at most $O(n^2)$ such quantum circuits we obtain the representation of any permutation in S_n. The Hadamard test allows a measurement to polynomial precision of the matrix elements of this representation.

8.3 Algorithm for Alternating Group

Any permutation π corresponds to a permutation matrix with matrix element i, j given by $\delta_{\pi(i),j}$. The determinant of any permutation matrix is ± 1, and is known as the sign of the permutation. The permutations of sign $+1$ are called even, and the permutations of sign -1 are called odd. This is because a transposition has determinant -1, and therefore any product of an odd number of transpositions is odd and any product of an even number of transpositions is even.

The even permutations in S_n form a subgroup called the alternating group A_n, which has size $n!/2$. A_n is a simple group (*i.e.* it contains no normal subgroup) and it is the only normal subgroup of S_n other than $\{\mathbb{1}\}$ and S_n. As one might guess, the irreducible representations of the alternating group are closely related to the irreducible representations of the symmetric group. Consequently, as shown in this section, the quantum algorithm of section 8.2 can be easily adapted to approximate any matrix element of any irreducible representation of A_n to within $\pm\epsilon$ in $\text{poly}(n, 1/\epsilon)$ time.

Explicit orthogonal matrix representations of the alternating group are worked out in [34] and recounted nicely in [23]. Any representation ρ of S_n is automatically also a representation of A_n. However an irreducible representation ρ of S_n may no longer be irreducible when restricted to A_n. Each irreducible representation of S_n either remains irreducible when restricted to A_n or decomposes into a direct sum of two irreducible representations of A_n. All of the irreducible representations of A_n are obtained in this way.

The conjugate of Young diagram λ is obtained by reflecting λ about the main diagonal, as shown in figure 4. If λ is not self-conjugate then the representation ρ_λ of S_n remains irreducible when restricted to A_n. In this case we can simply use the algorithm of section 8.2. If λ is self-conjugate then the representation ρ_λ of S_n becomes reducible when restricted to A_n. It is a direct sum of two irreducible representations of A_n, called $\rho_{\lambda+}$ and $\rho_{\lambda-}$. The two corresponding invariant subspaces of the reducible representation are the $+1$ and -1 eigenspaces, respectively, of the "associator" operator S defined as follows.

Let λ be a self-conjugate Young diagram of n boxes. Let Λ_0 be the "typewriter-order" Young tableau obtained by numbering the boxes from left to right across the first row, then left to right across the second row, and so on, as illustrated in figure 5. For any standard Young tableau Λ of shape λ, let $w_\Lambda \in S_n$ be the

1	2	3	4
5	6		
7			

Figure 5: For a given Young diagram, there is a unique Young tableau in "typewriter" order, in which the boxes are numbered from left to right across the top row then from left to right across the next row, and so on, as illustrated in the example above.

permutation that brings the boxes into typewriter order. That is, $w_\Lambda \Lambda = \Lambda_0$. Let $\hat{\Lambda}$ be the conjugate of Λ, obtained by reflecting Λ about the main diagonal. If Λ is standard then so is $\hat{\Lambda}$. Let $d(\lambda)$ be the length of the main diagonal of λ. S is the linear operator on $\mathcal{V}_\lambda$ defined by

$$S\Lambda = i^{(n-d(\lambda))/2} \text{sign}(w_\Lambda) \hat{\Lambda}. \tag{14}$$

An orthonormal basis for each of the eigenspaces of S can be easily constructed from the Young-Yamanouchi basis. When $(n - d(\lambda))/2$ is odd, every standard Young tableau Λ of shape λ has the property $\text{sign}(w_\Lambda) = -\text{sign}(w_{\hat{\Lambda}})$, and S is a direct sum of 2×2 blocks of the form

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

interchanging Λ and $\hat{\Lambda}$. In this case, the linear combinations $\frac{1}{\sqrt{2}}(\Lambda + i\hat{\Lambda})$ for each conjugate pair of standard Young tableaux form an orthonormal basis for the $+1$ eigenspace of S, and the linear combinations $\frac{1}{\sqrt{2}}(\Lambda - i\hat{\Lambda})$ form an orthonormal basis for the -1 eigenspace of S. Similarly, when $(n - d(\lambda))/2$ is even, $\text{sign}(w_\Lambda) = \text{sign}(w_{\hat{\Lambda}})$ for all standard Young tableaux Λ of shape λ. Thus S is a direct sum of 2×2 blocks of the form

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

interchanging Λ and $\hat{\Lambda}$. In this case the linear combinations $\frac{1}{\sqrt{2}}(\Lambda - \hat{\Lambda})$ form an orthonormal basis for the $+1$ eigenspace of S and the linear combinations $\frac{1}{\sqrt{2}}(\Lambda + \hat{\Lambda})$ form an orthonormal basis for the -1 eigenspace of S.

Suppose λ is self-conjugate and $(n-d(\lambda))/2$ is even. Any matrix element of the irreducible representation $\rho_{\lambda+}$ of A_n is given by

$$\frac{1}{2}(\Lambda + \hat{\Lambda})\rho_\lambda(\pi)(\Gamma + \hat{\Gamma}),$$

where Λ, Γ is some pair of standard Young tableaux and π is some element of A_n. This is a linear combination of only four Young-Yamanouchi matrix elements of $\rho_\lambda(\pi)$. One can use the algorithm of section 8.2 to calculate each of these and then simply add them up with the appropriate coefficients. The cases where $(n - d(\lambda))/2$ is odd and/or we want a matrix element of $\rho_{\lambda-}$ are analogous.

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A $\|a_{\vec{m}}(H_p)\|$ is independent of p

As shown in section 7.3, the irreducible representation of an arbitrary $u \in U(n)$ with weight $\vec{m}$ can be computed by simulating the time evolution according to a series of Hamiltonians of the form $A_{\vec{m}}(H_p)$, where $A_{\vec{m}}$ is the Gel'fand-Tsetlin representation of the Lie algebra $su(n)$ and

$$H_p = 0_p \oplus h \oplus 0_{n-p-2},$$

where h is a 2×2 antihermitian matrix. The quantum algorithm for simulating these Hamiltonians require that $\|A_{\vec{m}}(H_p)\|$ be at most poly(n). In section 7.3 we showed this to be the case for $p = 0$. Here we prove it for all p by showing:

Proposition 1 *Let h be a fixed 2×2 antihermitian matrix and let $H_p = 0_p \oplus h \oplus 0_{n-p-2}$. Let $a_{\vec{m}}$ be the Gel'fand-Tsetlin representation of $su(n)$ with weight $\vec{m}$. Then $\|a_{\vec{m}}(H_p)\|$ is independent of p.*

Proof:
Let $U_p^k = e^{kH_p}$. Then

$$U_p^k = \mathbb{1}_p \oplus e^{kh} \oplus \mathbb{1}_{n-p-2}.$$

Thus for any $0 \leq q \leq n$, there exists $V \in U(n)$ such that

$$U_q^k = V U_p^k V^{-1}. \tag{15}$$

Specifically, V is just a permutation matrix. Let $A_{\vec{m}}$ be the Gel'fand-Tsetlin representation of $SU(n)$. That is,

$$A_{\vec{m}}(U_q^k) = e^{a_{\vec{m}}(kH_q)}.$$

Thus

$$\begin{aligned} \left\| \frac{\mathrm{d}}{\mathrm{d}k} A_{\vec{m}}(U_p^k) \right\| &= \|a_{\vec{m}}(H_p) e^{k a_{\vec{m}}(H_p)}\| \\ &= \|a_{\vec{m}}(H_p)\|. \end{aligned} \tag{16}$$

Here we have used the fact that $A_{\vec{m}}$ is a unitary representation. Similarly,

$$\|a_{\vec{m}}(H_q)\| = \left\| \frac{\mathrm{d}}{\mathrm{d}k} A_{\vec{m}}(U_q^k) \right\|.$$

Using equation 15, this is equal to

$$\left\| \frac{\mathrm{d}}{\mathrm{d}k} A_{\vec{m}}(V U_p^k V^{-1}) \right\|.$$

Because $A_{\vec{m}}$ is a group homomorphism and V is independent of k this is equal to

$$\left\| A_{\vec{m}}(V) \left(\frac{\mathrm{d}}{\mathrm{d}k} A_{\vec{m}}(U_p^k) \right) A_{\vec{m}}(V)^{-1} \right\|.$$

Because $A_{\vec{m}}$ is a unitary representation this is equal to

$$\left\| \frac{\mathrm{d}}{\mathrm{d}k} A_{\vec{m}}(U_p^k) \right\|.$$

By equation 16 this is equal to $\|a_{\vec{m}}(H_p)\|$. $\square$

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